Lecture 9. Operations on matrices and linear transformations

Def Given an $m\times n$ matrix A and an $n\times \ell$ matrix B, their *product* AB is the matrix with columns $A\vec{b_1}, A\vec{b_2}, \cdots, A\vec{b_\ell}$ where $\vec{b_1}, \vec{b_2}, \cdots, \vec{b_\ell}$ are the columns of B.

e.g. $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 3 \\ 0 & -2 \end{bmatrix}$ with columns $\vec{b_1}$, $\vec{b_2}$

$$\Rightarrow A\vec{b_1} = \begin{bmatrix} 2\cdot 4 + 3\cdot 0 \\ 1\cdot 4 + 0\cdot 0 \\ 0\cdot 4 + (-1)\cdot 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}, \quad A\vec{b_2} = \begin{bmatrix} 2\cdot 3 + 3\cdot(-2) \\ 1\cdot 3 + 0\cdot(-2) \\ 0\cdot 3 + (-1)\cdot(-2) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} 8 & 0 \\ 4 & 3 \\ 0 & 2 \end{bmatrix}$$

BA is undefined ($B: 2\times 2$, $A: 3\times 2$) not equal

Note (1) The product AB is an $m\times \ell$ matrix.

(2) In general, AB and BA may be unequal even if both products are well-defined.

e.g. $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}$

$$\Rightarrow AB = \begin{bmatrix} 7 & -2 \\ 3 & -6 \end{bmatrix}, \quad BA = \begin{bmatrix} 6 & 3 \\ 2 & -5 \end{bmatrix}$$

(3) The product of two nonzero matrices may be zero.

e.g. $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Prop For arbitrary matrices A, B, C, the following identities hold as long as all relevant operations are well-defined.

(1) $A(BC) = (AB)C$

(2) $A(B+C) = AB + AC$

(3) $(A+B)C = AC + BC$

Note We can add matrices of the same size.

e.g. $\begin{bmatrix} 2 & 0 & 3 \\ 1 & -1 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 6 & -2 \\ 0 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 2+1 & 0+6 & 3-2 \\ 1+0 & -1+5 & 4+3 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 1 \\ 1 & 4 & 7 \end{bmatrix}$

Def A <u>square matrix</u> is a matrix with the same number of rows and columns.

e.g. $\underbrace{1\times 1 \text{ matrices}}_{\text{numbers}}$, 2×2 matrices, 3×3 matrices, $\cdots$

Note For a square matrix A, its powers are well-defined.

e.g. $A^2 = AA$, $A^3 = AAA$, $\cdots$

* In fact, all of its powers have the same size.

Thm Given linear transformations $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $S: \mathbb{R}^\ell \longrightarrow \mathbb{R}^n$ with standard matrices A and B, their composition $T \circ S$ is a linear transformation with standard matrix AB.

pf $T \circ S(\vec{x}) = T(S(\vec{x})) = T(B\vec{x}) = A(B\vec{x}) = AB\vec{x}$

Ex Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which first rotates each vector about the origin through $\frac{\pi}{6}$ radians then reflects each vector through the line $y = x$.

Sol T is the composition of the following linear transformations:

- the rotation T_1 about the origin through $\frac{\pi}{6}$ radians,
- the reflection T_2 through the line $y = x$.

The standard matrices of T_1 and T_2 are

$$A_1 = \begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

as seen in Lecture 6.

$\Rightarrow$ $\underline{T = T_2 \circ T_1}$ has standard matrix

The order is important!

$$\underline{A_2 A_1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \cdot \cos(\pi/6) + 1 \cdot \sin(\pi/6) & 0 \cdot (-\sin(\pi/6)) + 1 \cdot \cos(\pi/6) \\ 1 \cdot \cos(\pi/6) + 0 \cdot \sin(\pi/6) & 1 \cdot (-\sin(\pi/6)) + 0 \cdot \cos(\pi/6) \end{bmatrix}$$

$$= \begin{bmatrix} \sin(\pi/6) & \cos(\pi/6) \\ \cos(\pi/6) & -\sin(\pi/6) \end{bmatrix} = \boxed{\begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}}$$

Note We can get the same answer by directly computing the columns $T(\vec{e}_1)$ and $T(\vec{e}_2)$.

Appendix Sine/cosine summation formula

$$\begin{cases} \sin(\alpha+\beta) = \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta \\ \cos(\alpha+\beta) = \cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta \end{cases}$$

pf Consider the following linear transformations on $\mathbb{R}^2$:

- the rotation T_1 about the origin through α radians,
- the rotation T_2 about the origin through β radians.

The standard matrices of T_1 and T_2 are

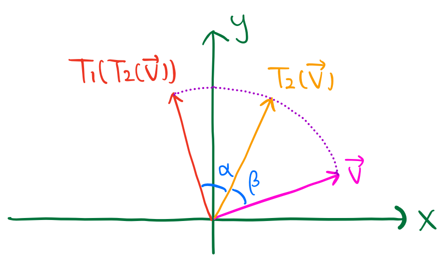
$$A_1 = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$\Rightarrow$ The standard matrix of $T_1 \circ T_2$ is

$$A_1A_2 = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta & -\cos\alpha\cdot\sin\beta + \sin\alpha\cdot\cos\beta \\ \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta & -\sin\alpha\cdot\sin\beta + \cos\alpha\cdot\cos\beta \end{bmatrix}$$

Moreover, $T_1 \circ T_2$ is the rotation about the origin through $\alpha+\beta$ radians.



$\Rightarrow$ The standard matrix of $T_1 \circ T_2$ is

$$A = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$$A = A_1A_2 \Rightarrow \begin{cases} \cos(\alpha+\beta) = \cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta \\ \sin(\alpha+\beta) = \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta \end{cases}$$